Due Sun

1.9 – Compositions of Matrix Transformations

The **composition of $\boldsymbol{T_B}$ with $\boldsymbol{T_A}$** is achieved by first applying the matrix transformation T_A to a vector and then applying the matrix transformation T_B to the image vector. We denote the composition of T_B with T_A by $T_B \circ T_A$ which is read "T_B circle T_A." This is also expressed as $(T_B \circ T_A)(\mathbf{x}) = T_B\big(T_A(\mathbf{x})\big)$.

Theorem 1.9.1 If $T_A: R^n \to R^k$ and $T_B: R^k \to R^m$ are matrix transformations, then $T_B \circ T_A$ is also a matrix transformation, and $T_B \circ T_A = T_{BA}$.

Rationale: $(T_B \circ T_A)(\vec{x}) = T_B(T_A(\vec{x}))$

$$= T_B(A\vec{x})$$

$$= B(A\vec{x})$$

$$= (\underbrace{B}_{m\times k}\underbrace{A}_{k\times n})\vec{x}$$

8. Find the standard matrix for the stated composition in R^2.

a. A rotation about the origin of $60°$, followed by an orthogonal projection onto the x-axis, followed by a reflection about the line $y = x$.

b. An orthogonal projection onto the x-axis, followed by a rotation about the origin of $45°$, followed by a reflection about the y-axis.

c. A rotation about the origin of $15°$, followed by a rotation about the origin of $105°$, followed by a rotation about the origin of $60°$.

a. $R_{60^\circ} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$B\,A\,R_{60^\circ} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 1/2 & -\sqrt{3}/2 \end{bmatrix}$$

$$T_1\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} \underline{4x_1} \\ \underline{-2x_1 + x_2} \\ \underline{-x_1 - 3x_2} \end{bmatrix}$$

12. Let $T_1(x_1, x_2, x_3) = (\underline{4x_1}, \underline{-2x_1 + x_2}, \underline{-x_1 - 3x_2})$ and $T_2(x_1, x_2, x_3) = (x_1 + 2x_2, -x_3, 4x_1 - x_3)$.

a. Find the standard matrices for T_1 and T_2.

b. Find the standard matrices for $T_2 \circ T_1$ and $T_1 \circ T_2$.

c. Use the matrices obtained in part (b) to find formulas for $T_1(T_2(x_1, x_2, x_3))$ and $T_2(T_1(x_1, x_2, x_3))$.

a. $[T_1] = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -3 & 0 \end{bmatrix}$, $[T_2] = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 4 & 0 & -1 \end{bmatrix}$

$$b.\ [T_2 \circ T_1] = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 4 & 0 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 & 0 \\ 1 & 3 & 0 \\ 17 & 3 & 0 \end{bmatrix} \leftarrow \text{matrix}$$

formula

$$c.\ T_2(T_1(x_1, x_2, x_3)) = (2x_2,\ x_1+3x_2,\ 17x_1+3x_2)$$

If $T_A: R^n \to R^n$ is a matrix operator whose standard matrix A is invertible, then T_A is **invertible**, and the **inverse** of T_A is $T_A^{-1} = T_{A^{-1}}$.

20. Determine whether the matrix operator $T: R^3 \to R^3$ defined by the equations is invertible; if so, find the standard matrix for the inverse operator, and find $T^{-1}(w_1, w_2, w_3)$.

a.

$w_1 = x_1 - 2x_2 + 2x_3$
$w_2 = 2x_1 + x_2 + x_3$
$w_3 = x_1 + x_2$

b.

$w_1 = x_1 - 3x_2 + 4x_3$
$w_2 = -x_1 + x_2 + x_3$
$w_3 = -2x_2 + 5x_3$